UNIVERSITY OF CALGARY
HASKAYNE SCHOOL OF BUSINESS

Investments & Portfolio Management

Optimal Risky Portfolios and Diversification

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Having addressed the issue of the optimal allocation between the risk-free security and a portfolio of risk securities, the next step consists in finding the optimal portfolio of risky securities. In practice, the first key decision is the allocation between classes of risky assets (equities, bonds, etc.) and the second step consists in the selection of individual securities within each asset class. Of note, nowadays the availability of cost-efficient ETFs for most asset classes offers a ready-made solution to bypass the selection of individual securities.

One key analytical driver of the optimization of a portfolio of risky security is to use diversification to reduce risk without reducing expected return too much. This requires a careful assessment of the expected correlations between asset classes and between each security within a given asset class. While these correlations vary through time, they are nevertheless thought to be somewhat more stable and predictable than returns.

For a portfolio of N securities, the return r_p and the risk σ_p are respectively:

$$r_p = \sum_{i=1}^{N} w_i r_i \leftrightarrow Er_p = \sum_{i=1}^{N} w_i Er_i, \qquad \sum_{i=1}^{N} w_i = 1$$

$$\sigma_p^2 = Var\left(r_p\right) = Var\left[\sum_{i=1}^{N} w_i r_i\right] = \sum_{i,j=1}^{N} w_i w_j Covar\left(r_i, r_j\right) = \sum_{i=1}^{N} w_i^2 Var\left(r_i\right) + \sum_{i,j=1,i\neq j}^{N} w_i w_j Covar\left(r_i, r_j\right)$$

For a portfolio of two securities, the return r_p and the risk σ_p are respectively:

$$r_p = w_1\, r_1 + \left(1 - w_1\right) r_2 \leftrightarrow Er_p = w_1\, Er_1 + \left(1 - w_1\right) Er_2$$

$$\sigma_p^2 = w_1^2 Var\left(r_1\right) + w_2^2 Var\left(r_2\right) + 2 w_1 w_2 Covar\left(r_1, r_2\right) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{12} \sigma_1 \sigma_2$$

Derivation of the variance of a portfolio of two assets

Variance of a portfolio of two assets using $Var(X) = E(X - EX)^2$:

$$
\begin{aligned}
E(w_x X + w_Y Y - w_x EX - w_Y EY)^2 &= E(w_x (X - EX) + w_Y (Y - EY))^2 \\
&= E\left(w_X^2 (X - EX)^2 + w_Y^2 (Y - EY)^2 + 2 w_X w_Y (X - EX)(Y - EY)\right) \\
&= w_X^2 E(X - EX)^2 + w_Y^2 E(Y - EY)^2 + 2 w_X w_Y E(X - EX)(Y - EY) \\
&= w_X^2 Var(X) + w_Y^2 Var(Y) + 2 w_X w_Y Cov(X, Y) \\
&= w_X^2 \sigma_X^2 + w_Y^2 \sigma_Y^2 + 2 w_X w_Y \rho_{XY} \sigma_X \sigma_Y
\end{aligned}
$$

- The relationship between the variance of the return of a portfolio and the variance of the returns of its individual components is **not linear**.

Covariance matrix of two assets and matrix (vector) of portfolio weights

$$\boldsymbol{V} = \begin{bmatrix} (X-EX)^2 & (Y-EY)(X-EX) \\ (X-EX)(Y-EY) & (Y-EY)^2 \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

Variance of the portfolio (conformable: [1x2][2x2][2x1]=[1x2][2x1]=[1x1] yields a scalar)

$$\sigma_p^2 = \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{V} \boldsymbol{w} = \begin{bmatrix} w_x & w_y \end{bmatrix} \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

$$= \begin{bmatrix} w_x & w_y \end{bmatrix} \begin{bmatrix} w_x \sigma_X^2 + w_y \sigma_{XY} \\ w_x \sigma_{XY} + w_y \sigma_Y^2 \end{bmatrix} = w_x w_x \sigma_X^2 + w_x w_y \sigma_{XY} + w_y w_x \sigma_{XY} + w_y w_y \sigma_Y^2$$

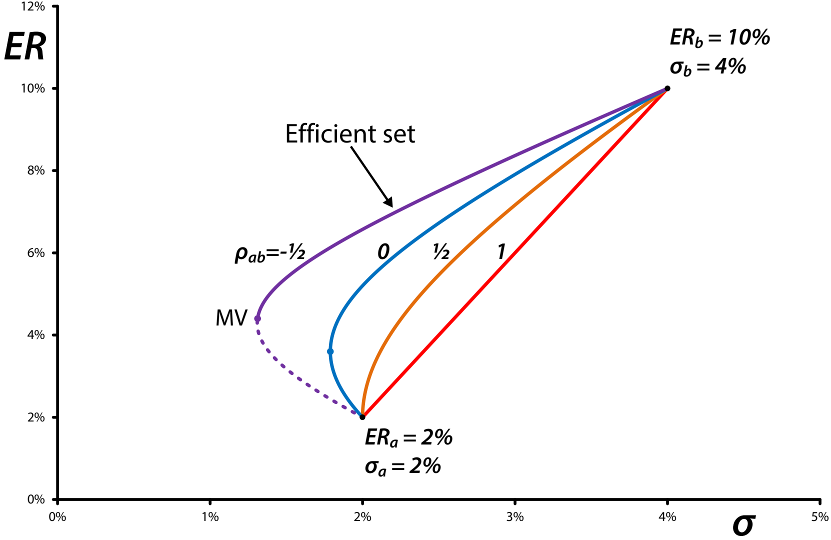
Revise your matrix algebra and do concept check 7.1 of the textbook.

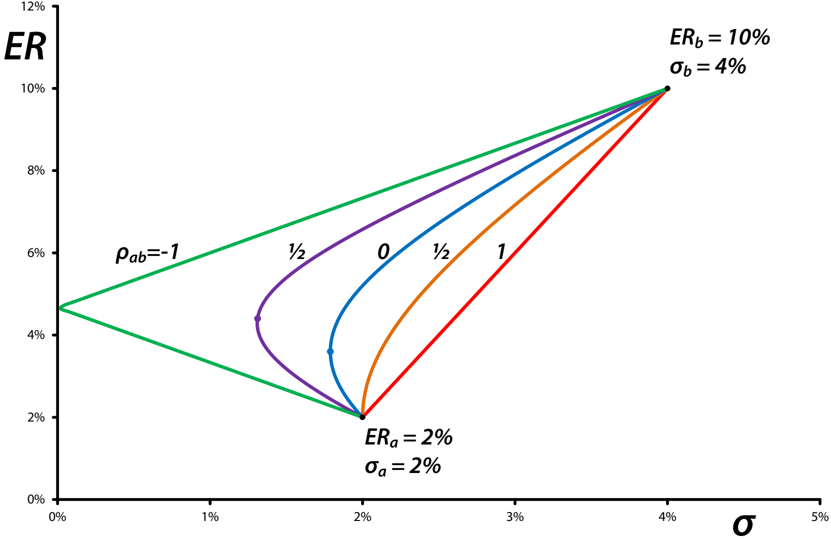
A correlation coefficient between two securities (e.g. ρ_{12}) ranges from -1 to 1.

When $\rho_{12} = 1$ the returns are perfectly positively correlated (i.e. securities 1 and 2 'move together' 100% of the time). This precludes diversification since $\sigma_p = w_1\sigma_1 + w_2\sigma_2$.

When $\rho_{12} = 0$ the returns are perfectly uncorrelated (i.e. securities 1 and 2 'move independently' from one another 100% of the time). This allows for diversification since $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$.

When $\rho_{12} = -1$ the returns are perfectly negatively correlated (i.e. securities 1 and 2 'move in opposite directions' 100% of the time). This allows for perfect diversification using $w_1 = \frac{\sigma_2}{\sigma_1+\sigma_2} = 1 - w_2$.


12%
ER
10%
8%
6%
4%
2%
0%
$ER_b = 10\%$
$\sigma_b = 4\%$
Efficient set
ρ_{ab}=-½
0
½
1
MV
$ER_a = 2\%$
$\sigma_a = 2\%$
0%
1%
2%
3%
4%
5%
σ


ER
12%
10%
8%
6%
4%
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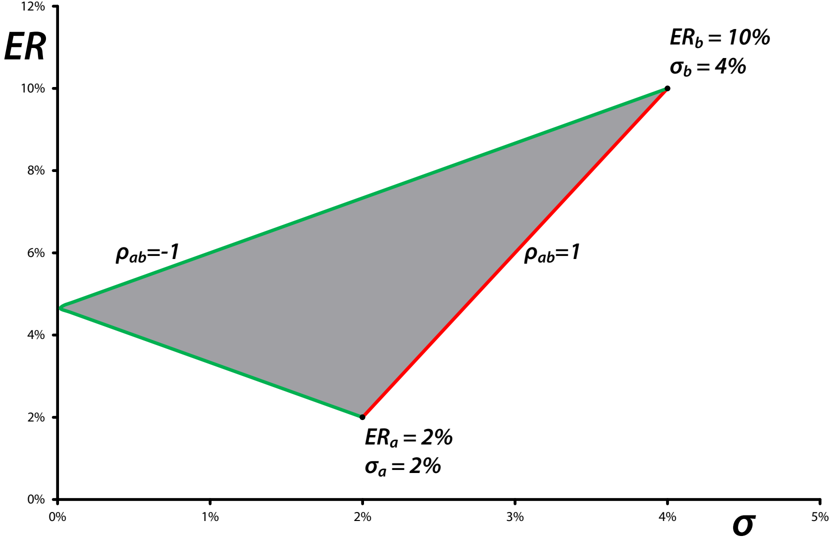
0%
0%
1%
2%
3%
4%
5%
σ
$ER_b = 10\%$
$\sigma_b = 4\%$
$ER_a = 2\%$
$\sigma_a = 2\%$
ρ_{ab}=-1
½
0
½
1



12%
ER
10%
8%
6%
4%
2%
0%
0%
1%
2%
3%
4%
5%
σ
$ER_b = 10\%$
$\sigma_b = 4\%$
ρ_{ab}=-1
ρ_{ab}=1
$ER_a = 2\%$
$\sigma_a = 2\%$

Assumptions for a portfolio of many risky assets

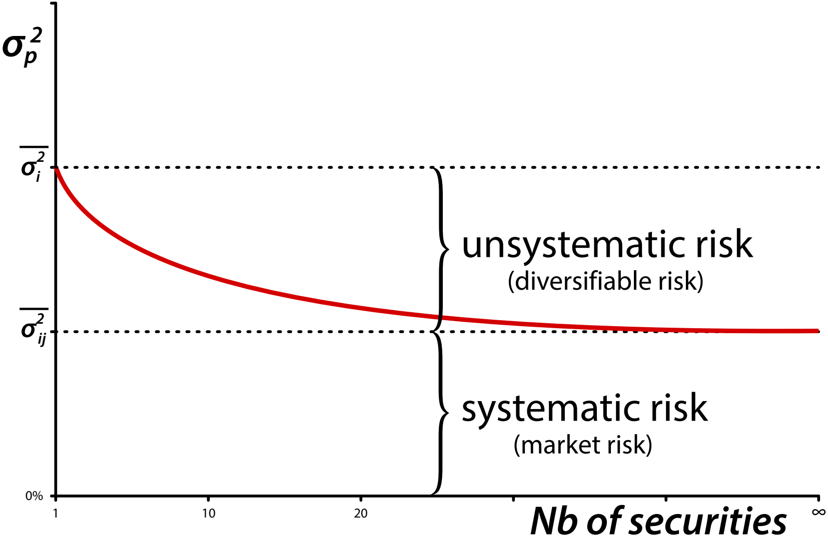
- All securities have the same variance, denoted $\overline{\sigma_i^2}$.
- All covariances are identical, denoted $\overline{\sigma_{ij}^2}$ (i.e. same $\overline{\rho_{ij}\sigma_i\sigma_j}$).
- All securities are equally weighted w_i.

$$\sigma_p^2 = \left(\frac{1}{N}\right)\overline{\sigma_i^2} + \left(1 - \frac{1}{N}\right)\overline{\sigma_{ij}^2} = \overline{\sigma_{ij}^2} \quad as\ N \to \infty$$

As N becomes larger, the first term, the contribution of individual securities to the risk of the portfolio, becomes increasingly small but the second term approaches the average covariance.

The variance of such portfolio is **not lower than the average covariance** of the component securities and the variance of each security does not matter much.

Decreasing the risk of a well-diversified portfolio of risky assets requires finding new securities with fairly low correlations with the existing portfolio (the marketing pitch of hedge funds).


σ_p^2
$\overline{\sigma_i^2}$
$\overline{\sigma_{ij}^2}$
0%
1
10
20
∞
Nb of securities
unsystematic risk
(diversifiable risk)
systematic risk
(market risk)

Correlations and volatility

Other asset classes

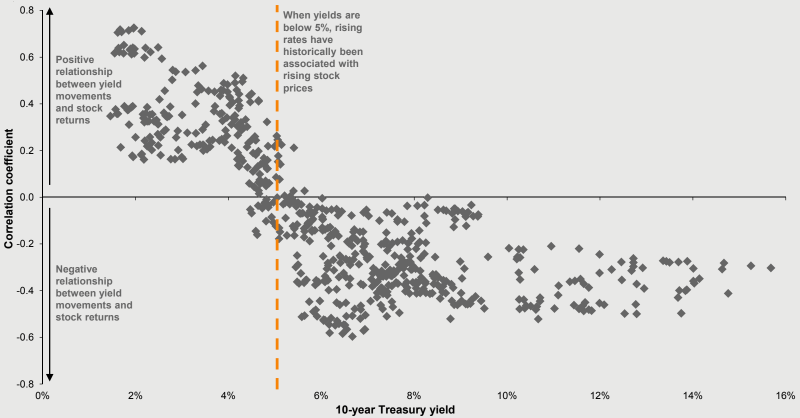
	U.S. Large Cap	EAFE	EME	Bonds	Corp. HY	Munis	Currcy.	EMD	Cmdty.	REITs	Hedge funds	Private equity	Ann. Volatility
U.S. Large Cap	1.00	0.89	0.79	-0.29	0.75	-0.10	-0.45	0.62	0.53	0.79	0.82	0.84	16%
EAFE		1.00	0.91	-0.13	0.80	0.02	-0.64	0.73	0.60	0.68	0.86	0.82	20%
EME			1.00	-0.01	0.88	0.12	-0.68	0.85	0.67	0.59	0.86	0.78	24%
Bonds				1.00	-0.06	0.80	-0.23	0.26	-0.11	0.00	-0.19	-0.25	3%
Corp. HY					1.00	0.11	-0.51	0.88	0.66	0.67	0.82	0.72	12%
Munis						1.00	-0.22	0.45	-0.09	0.07	0.01	-0.13	4%
Currencies							1.00	-0.62	-0.62	-0.40	-0.47	-0.55	8%
EMD								1.00	0.59	0.60	0.71	0.62	8%
Commodities									1.00	0.40	0.72	0.71	21%
REITs										1.00	0.56	0.66	25%
Hedge funds											1.00	0.86	7%
Private equity												1.00	11%

Source: Barclays Inc., Bloomberg, Cambridge Associates, Credit Suisse/Tremont, FactSet, Federal Reserve, MSCI, NCREIF, Standard & Poor's, J.P. Morgan Asset Management.
Indexes used – Large Cap: S&P 500 Index; Currencies: Federal Reserve Trade Weighted Dollar; EAFE: MSCI EAFE; EME: MSCI Emerging Markets; Bonds: Barclays Aggregate; Corp HY: Barclays Corporate High Yield; EMD: Barclays Emerging Market; Cmdty.: Bloomberg Commodity Index; Real Estate: NAREIT ODCE Index; Hedge Funds: CS/Tremont Hedge Fund Index; Private equity: Cambridge Associates Global Buyout & Growth Index. Private equity data are reported on a two quarter lag. All correlation coefficients and annualized volatility calculated based on quarterly total return data for period 6/30/07 to 6/30/17. This chart is for illustrative purposes only.
Guide to the Markets – U.S. Data are as of June 30, 2017.

J.P.Morgan Asset Management

Equities

Correlations between weekly stock returns and interest rate movements
Weekly S&P 500 returns, 10-year Treasury yield, rolling 2-year correlation, June 1963 – June 2017



Source: FactSet, Standard & Poor's, FRB, J.P. Morgan Asset Management.
Returns are based on price index only and do not include dividends. Markers represent monthly 2-year correlations only.
Guide to the Markets – U.S. Data are as of June 30, 2017.

J.P.Morgan
Asset Management

Potential sources of diversification

Diversify across asset classes

- As already shown correlation between asset classes if often low, even negative.

Diversify with index funds (e.g. ETFs)

- Diversification can be costly, but using index funds can be very cost efficient.

International diversification

- Investment in foreign securities can significantly diversify a domestic-focused portfolio.

Do not over-invest in your employer's stock or industry

- Portfolio should not be correlated with cash flow from employment (human capital).

Evaluate the risk-return trade-off when adding a new asset to an existing portfolio

- OK if Sharpe ratio of the new asset > Sharpe ratio of portfolio x correlation coefficient.

Insure the portfolio

- Invest in assets having a negative correlation with the existing portfolio, e.g. put options.

Given the data on the three available securities, the optimal weights of the two risky securities are the weights that provide for the maximum Sharpe ratio.

$$\underset{w}{Max}\; S_p = \frac{Er_p - r_f}{\sigma_p} \quad s.t \sum w_i = 1$$

(... we skip many steps ...)

$$\boldsymbol{W} = \frac{\boldsymbol{V}^{-1}\boldsymbol{\xi}}{\boldsymbol{1}^T\boldsymbol{V}^{-1}\boldsymbol{\xi}} = \frac{\frac{1}{x}\,[2x2]\,[2x1]}{[1x2]\,\frac{1}{x}\,[2x2]\,[2x1]} = \frac{[2x1]}{[1x1]} = [2x1] = \begin{bmatrix} w_B \\ w_S \end{bmatrix}$$

$$\boldsymbol{V} = \begin{bmatrix} \sigma_B^2 & \sigma_{SB} \\ \sigma_{BS} & \sigma_S^2 \end{bmatrix} \quad \boldsymbol{V}^{-1} = \frac{1}{\sigma_S^2\sigma_B^2 - \sigma_{BS}^2}\begin{bmatrix} \sigma_S^2 & -\sigma_{BS} \\ -\sigma_{SB} & \sigma_B^2 \end{bmatrix} \quad \boldsymbol{\xi} = \begin{bmatrix} Er_B - r_f \\ Er_S - r_f \end{bmatrix} \quad \boldsymbol{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$w_B = \frac{[Er_B - r_f]\,\sigma_S^2 - [Er_S - r_f]\,\sigma_{BS}}{[Er_B - r_f]\,\sigma_S^2 + [Er_S - r_f]\,\sigma_B^2 - [Er_B - r_f + Er_S - r_f]\,\sigma_{BS}} \qquad w_S = 1 - w_B$$

Let's construct an optimal asset allocation with stocks, bonds, and the risk free asset using the following actual **1984 to 2013** monthly data which we will use as proxy for our forecast of the future.

	Monthly (actual)		Annualized		Sharpe ratio
	Return	Std. Deviation	Return	Std. Deviation	
r_f	0.003241	0	4.0%	0	
10Y T-Bonds	0.006612	0.021664	8.2%	7.5%	0.57
S&P500	0.009901	0.044242	12.5%	15.3%	0.56
CPI	0.002322	0.003170	2.8%	1.1%	

Covariance and Correlation Matrices (hint: correlation matrices are easier to understand)

	r_f	T-Bonds	S&P500	CPI
r_f	4.8e-06			
10Y T-Bonds	6.7e-06	.000469		
S&P500	3.4e-06	.000015	.001957	
CPI	1.4e-06	-.00001	-3.1e-06	.00001

	r_f	T-Bonds	S&P500	CPI
r_f	1.0000			
10Y T-Bonds	0.1412	1.0000		
S&P500	0.0345	0.0157	1.0000	
CPI	0.1981	-0.1501	-0.0219	1.0000

Using the inputs x100 into the formula we find bonds should be 67.88% of the portfolio and 32.12% for equity.

$$w_B = \frac{[0.6612 - 0.3241]\, 19.5735 - [0.9931 - 0.3241]\, 0.150355}{[0.6612 - 0.3241]\, 19.5735 + [0.9931 - 0.3241]\, 4.69333 - [0.6612 - 0.3241 + 0.9931 - 0.3241]\, 0.150355} = 0.6788$$

$$w_S = 1 - 0.6788 = 0.3212$$

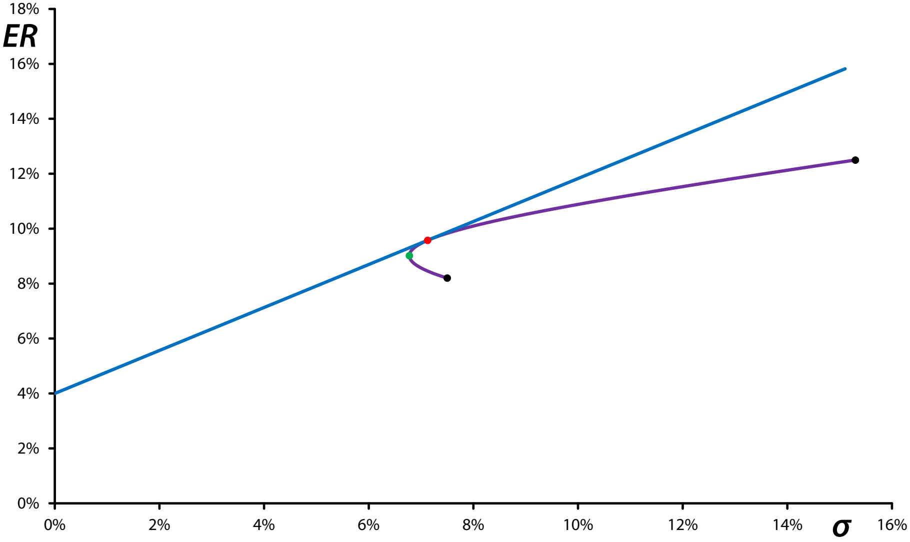
We also find that the expected monthly return is 0.7668% and its standard deviation is 2.061%. Once annualized, the return is 9.60% and its standard deviation is 7.14%, for a Sharpe ratio of 0.79.

$$Er_p = w_B Er_B + w_S Er_S = 0.007668 \quad \sigma_p = \sqrt{w_B^2 \sigma_B^2 + w_S^2 \sigma_S^2 + 2 w_B w_S \sigma_{BS}} = 0.02061$$

Assuming our investor has a risk tolerance $A = 4$, it is suggested to invest in an highly leveraged portfolio (?).

$$y^* = \frac{Er_p - r_f}{A\,\sigma_p^2} = \frac{0.007668 - 0.003241}{2 \times 0.02061^2} = 2.61$$

Asset allocation with stocks, bonds, and the risk-free asset



How our risky portfolio would have performed the next five years (actual **2014 to 2018** monthly data)?

	Monthly (actual)		Annualized		Sharpe ratio
	Return	Std. Deviation	Return	Std. Deviation	
r_f	0.0004494	0	0.5%	0	
10Y T-Bonds	0.0024882	0.0151477	3.0%	5.2%	0.47
S&P500	0.00728	0.031538	9.1%	10.9%	0.78
CPI	0.0012572	0.0029302	1.5%	1.0%	

The monthly correlation between T-Bonds and the S&P500 was -0.3087 for the period.

$$r_p = w_B r_B + w_S r_S = 0.004027 \quad \sigma_p = \sqrt{w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S \rho_{BS}\sigma_B\sigma_S} = 0.000144$$

The annualized return was 4.9% while the annualized standard deviation was 4.2%, for A Sharpe ratio of 1.06 (see Excel file). The portfolio return was lower than expected since the returns for both assets were lower than in the previous 30-year period. But, the risk was much lower since both assets were less risky, and the correlation was negative rather than slightly positive. In retrospect and with the benefit of hindsight, leveraging the portfolio would have been beneficial.

What we have seen is an application of the Markovitz portfolio optimization model with two risky assets and the risk-free rate. It could also be used in the same manner for multiple risky assets. For example, rather than using the S&P500 index, one could consider investing in each of its 500 constituent stocks. The same goes on with fixed income as bonds of various maturities are available, not only 10-year bonds. There is also non-US assets and alternative US assets classes like real-estate investment trusts.

Using many risk assets could help achieve a higher Sharpe ratio for the risky portfolio via increased diversification. But when the risk-free asset is available, we would end up using the same two-step process, i.e. choosing the weights of each risk security to achieve the risk portfolio that provides for the higher slope CAL (i.e. highest Sharpe ratio), and then adjusting the riskiness of the overall portfolio via the weight of the risk-free asset and the weight of the risk portfolio.

Of note, as the number of risky assets become large, for example in excess of 10 or 50, the mathematics to be used will have to change..

Applications in chapter 7 of the textbook

Concept checks

- Suggest to do concept checks 1 to 5 (solutions provided at the end of the chapter).

Exercises

- Suggest 7-12, 7-16, 7-17, 7-18, and 7-19.
- Solutions follow next slides, and Excel solution file in available in D2L.

7-12 Solution

Stock	ER	SD ER
A	10%	5%
B	15%	10%
Correlation	-1	

A risk-free portoflio of A and B would have a standard deviation of 0.

$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_A\sigma_B\rho_{AB}$

$\sigma_p^2 = w_A^2\sigma_A^2 - 2w_Aw_B\sigma_A\sigma_B + w_B^2\sigma_B^2$

$\sigma_p^2 = (w_A\sigma_A - \sigma_B w_B)^2$

σ_p =ABS$(w_A\sigma_A - w_B\sigma_B)$

0= $w_A 0.05 - (1 - w_A)0.10$

w_A =2/3 and w_B =1/3

ERf = 3/2x10% + 1/3*15% = 11.667%

7-16 Solution

	ER	SD ER
Risk free	12%	0%
Risky asset	30%	40%

A risk-free return has a colleration of 0 with the return of the risky asset.

$$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_A \sigma_B \rho_{AB}$$

$$\sigma_p^2 = w_R^2\sigma_R^2$$

$$\sigma_p = w_R\sigma_R \rightarrow w_R = \frac{\sigma_p}{\sigma_R} = \frac{0.3}{0.4} = 0.75$$

ERp = 0.25x12% + 0.75x30% = 25.5%

	ER	SD ER		Correlation
Stock A	8%	20%	A-B	0.85
Stock B	8%	20%	A-C	0.60
Stock C	8%	20%	A-D	0.45
Stock D	8%	20%		
Risk free	8%	0%		

17 When all stocks have the same expected rate of return, the optimal portfolio for any risk-averse investor is the global minimum variance portfolio (G). Given that all stocks have the same standard deviation of returns, the desired addition would be the stock with the lowest correlation with Stock A, which is Stock D.

18 No, as long as investors are not risk lovers while risk-neutral would not care which portfolio they held since all portfolios have an expected return of 8%.

19 Yes, the efficient frontier of risky assets is horizontal at 8%, so the optimal CAL runs from the risk-free rate through G: risk-averse investors will just hold Treasury bills.