UNIVERSITY OF CALGARY
HASKAYNE SCHOOL OF BUSINESS

Investments & Portfolio Management

Term Structure and Interest Rate Risk

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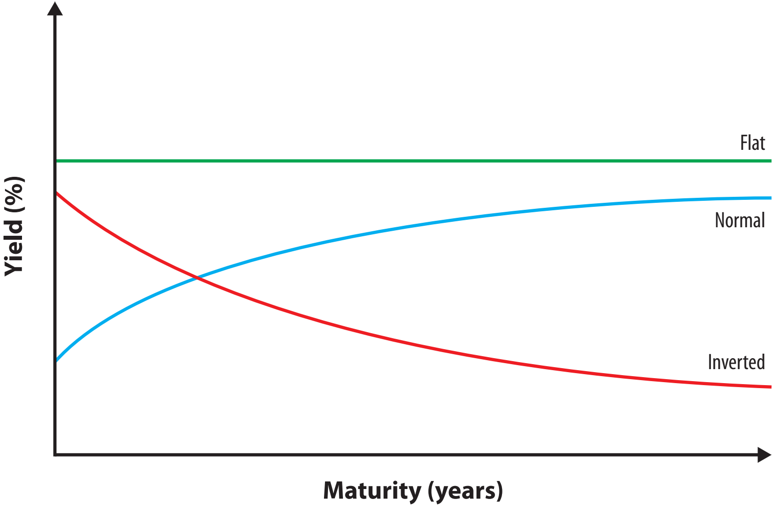
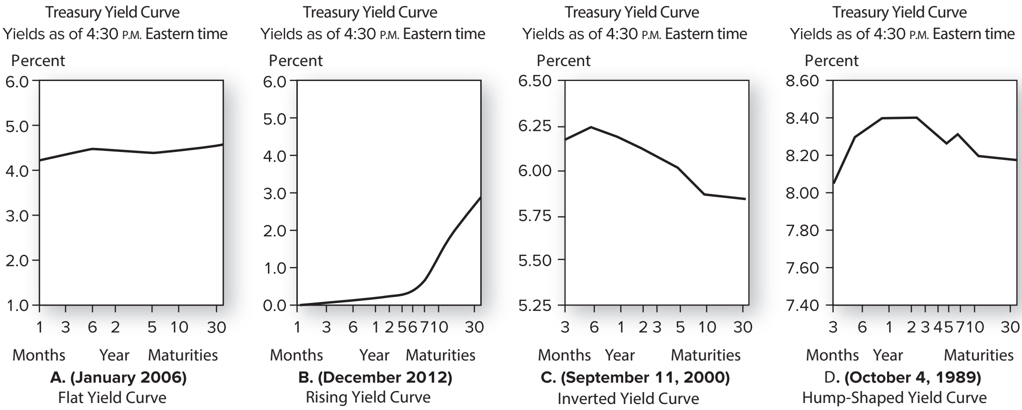
'The term structure of interest rates refers to the interest rates for various terms to maturity embodied in the prices of default-free zero-coupon bonds'. (the price of money is not constant)

- Given the market prices of default-free zero-coupon bonds at one point in time (i.e. a single cash-flow to be received in the future), it is possible to observe that interest rates vary according to how distant in the future such bonds mature;
- So, it is a simplification to use the same constant interest rate to discount all cash flows of a fixed-income security regardless when a given cash flow is to be received in the future;
- The 'term structure of interest rates' is the structure of interest rates used in the real world for discounting cash flows of different maturities, and it changes through time to reflect the market participants evolving expectations about the future;
- As the term structure of interest rates changes through time, it has a major impact on the prices of fixed-income securities and therefore their holding period returns;
- So, to understand fixed income securities, and be able to manage such portfolios, requires understanding the term structure of interest rates, and this requires some analytics.

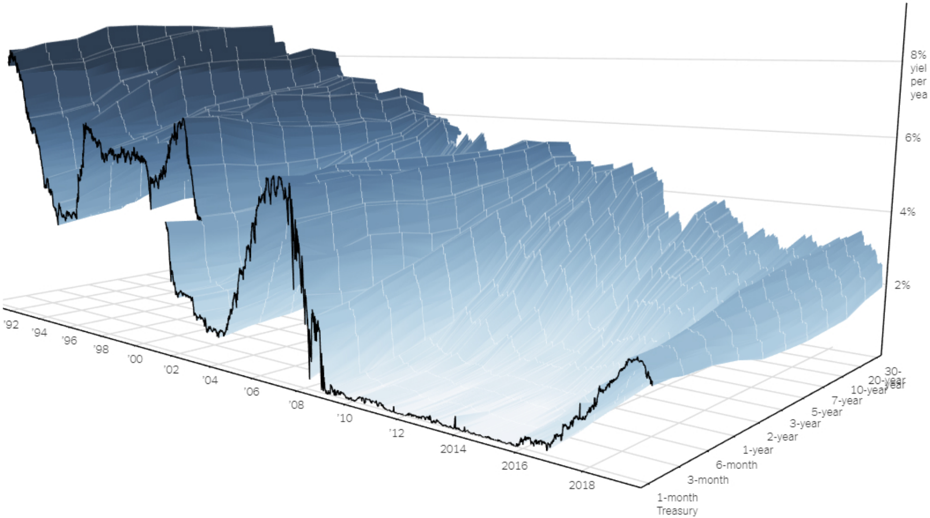
Typology of yield curves



Treasury yield curves

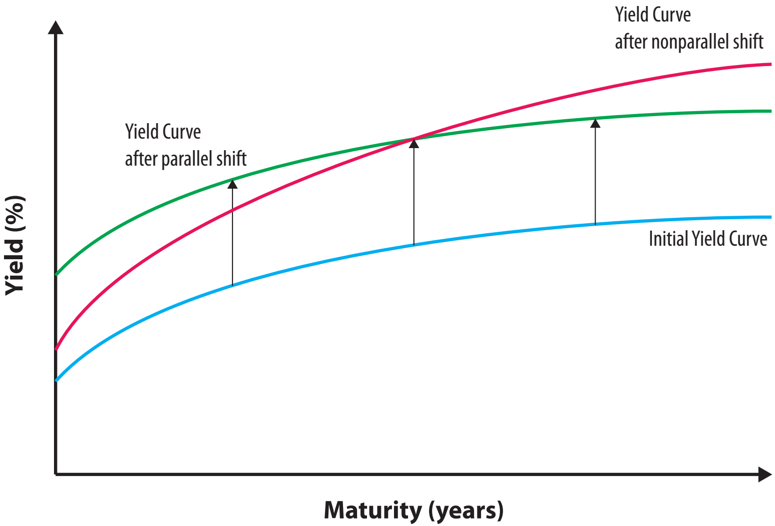


Treasury yield curves (click on the hyperlink and have a look)



source: https://www.nytimes.com/2019/08/15/upshot/inverted-yield-curve-bonds-football-analogy.html

Shifts in yield curves can be parallel or not



Extract the spot rates for different maturities using the prices of zero-coupon bonds.

Maturity	Price	Face value	Spot rate
1 year	\$952.38	\$1,000.00	$0.05 = \left(\frac{1000.00}{952.38}\right)^{1/1} - 1$
2 years	\$890.00	\$1,000.00	$0.06 = \left(\frac{1000.00}{890.00}\right)^{1/2} - 1$
3 years	\$816.30	\$1,000.00	$0.07 = \left(\frac{1000.00}{816.30}\right)^{1/3} - 1$

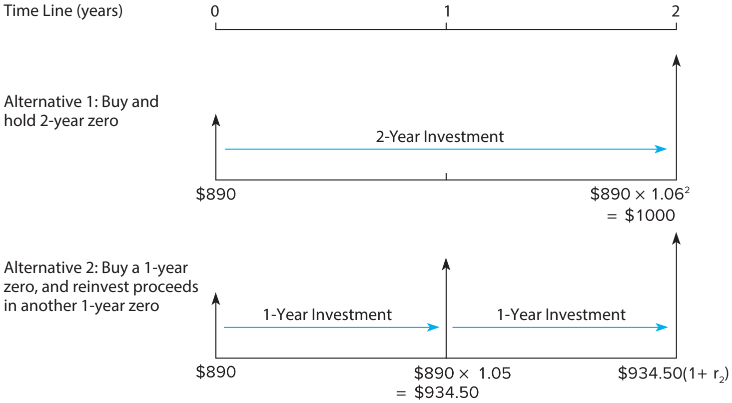
Value a coupon bond using the spot rates (e.g. 10% coupon with a maturity of three years).

$$P = \sum_{n=1}^{3} \frac{CF_n}{(1+r_n)^n} = \frac{100}{1.05^1} + \frac{100}{1.06^2} + \frac{1{,}100}{1.07^3} = 95.24 + 89.00 + 897.93 = 1{,}082.17$$

The value of the bond should be equal to the total value of its constituent parts.

- If not, then arbitrage by way of bond stripping or bond reconstitution.

For an investment horizon of n years, you can either buy a bond of n years **or** buy a bond of less than n years and 'roll over' at maturity by buying a second bond with the proceeds of the first until you reach n years.



In a world without uncertainty

Buy and hold = Roll-over

$P \times (1+y_2)^2 = P \times (1+r_1) \times (1+r_2)$

$(1+r_2) = (1+y_2)^2 \,/\, (1+r_1)$

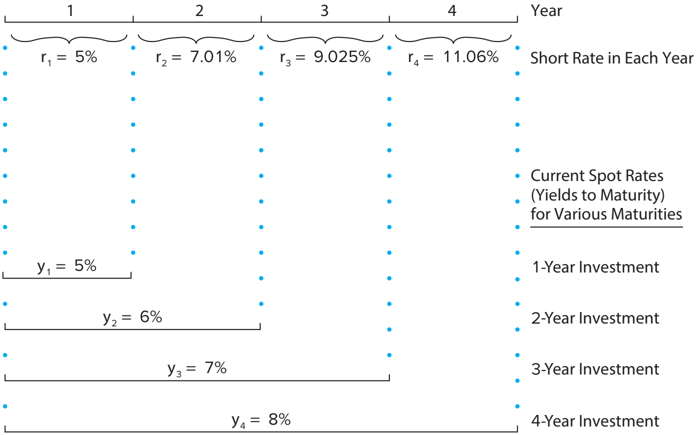
$r_2 = 1.06^2/1.05 - 1 = 7.01\%$

The 'short rate' for a given time interval refers to the interest rate for that interval available at different points in time.

A spot rate is the geometric average of its component short rates.

$(1+y_2) = \sqrt{(1+r_1) \times (1+r_2)}$

Given spot rates of various maturities, we can extract future various short rates which might not be the interest rate that actually prevail at future dates. Contractually agreed future rates are called forward interest rates.



$$(1+y_n)^n = (1+y_{n-1})^{n-1} \times (1+r_n)$$

In an uncertain world, the actual r_n in n years might not be the expected r_n. But given that the YTM y_n and y_{n-1} are both 'investable' (i.e. available), you can devise a contract to obtain r_n in n years called a forward (i.e. f_n).

$$(1+f_n) = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}}$$

$$(1+f_4) = \frac{(1+y_4)^4}{(1+y_3)^3} = \frac{1.08^4}{1.07^3} = 1.1106$$

	T=0	T=3	T=4
Buy 3-year zero bond	$ 816.30	$ 1,000.00	
Short 4-year zero bond	-$ 816.30		-$ 1,110.57
Lend to client		-$ 1,000.00	
Client repay			$ 1,110.57
Net	$ -	$ -	$ -

Extract the forward rates for different maturities using the spot rates.

Maturity	Price	Face value	Spot rate	Forward rate
1 year	\$952.38	\$1000.00	$0.05 = \left(\frac{1000.00}{952.38}\right)^{1/1} - 1$	0.05
2 years	\$890.00	\$1000.00	$0.06 = \left(\frac{1000.00}{890.00}\right)^{1/2} - 1$	$0.0701 = \frac{1.06^2}{1.05^1} - 1$
3 years	\$816.30	\$1000.00	$0.07 = \left(\frac{1000.00}{816.30}\right)^{1/3} - 1$	$0.0903 = \frac{1.07^3}{1.06^2} - 1$

Value a coupon bond using the forward rates (e.g. 10% coupon with a maturity of three years).

$$P = \frac{100}{1.05} + \frac{100}{1.05 \times 1.0701} + \frac{1{,}100}{1.05 \times 1.0701 \times 1.0903} = 95.24 + 89.00 + 897.93 = 1{,}082.17$$

Buying and holding a risk-free bond of n years until maturity will deliver y_n, its initial yield to maturity. But in the real world, a world with uncertainty, replicating such buy-and-hold using a series of rollovers is risky since the actual short rates are likely to differs from the initially calculated short rates.

- For example, getting y_2 through a rollover under $(1+y_2) = (1+r_1) \times (1+r_2)$ will only work if the actual r_2 turns out to be exactly equals to the initially expected r_2, but this is unlikely.

Conversely, in the real world, a world with uncertainty, buying a risk-free bond of more than n years and selling it after n years will likely not deliver y_n, since, again, the actual short rates are likely to differs from the initially calculated short rates.

- For example, investing in a 2-year bond and selling it after one year is unlikely to deliver r_1 under $(1+r_1) = \frac{(1+y_2)}{(1+r_2)}$ since it will only occur if the actual r_2 turns out to be exactly equal to the initially expected r_2, but this is unlikely.

Furthermore, the longer the bond maturity, the greater the price change induced by a given change in interest rate. So, risk-adverse short-term investors require a 'liquidity premium' to be compensated for bearing such risk (i.e. uncertain return if they sell a long-term bond early). This is assumed to explain why most of the time an upward sloped term structure of interest rate is observed (i.e. $f_n > Er_n$).

The Expectations Hypothesis Theory

- Assume forward rates are unbiased predictors of future short interest rates (i.e. $f_n = Er_n$).
- Then $(1 + y_2) = (1 + r_1) \times (1 + f_2) = (1 + r_1) \times (1 + Er_2)$.
- An upward-sloping yield curve indicate investors are expecting increases in interest rates.
- Conflicts with the evidence that investors are on average risk-adverse.

The Liquidity Preference Theory (unrelated to a liquidity premium required for thinly traded securities)

- Assume short-term investors require a premium to hold long-term bonds (e.g. $f_n > Er_n$) and, conversely, long-term investors require a premium to hold short-term bonds (e.g. $f_n < Er_n$), but short-term investors dominate the market (therefore $f_n > Er_n$).
- A slightly upward-sloping yield curve could indicate investors are expecting stable interest rates but require a term premium.
- A strongly upward-sloping yield curve could indicate investors are expecting increases in interest rates while also requiring a term premium.
- Consistent with the assumed preference of issuers to issue long-term to lock-in interest rates and therefore be willing to pay some premium to do so.

Interpreting term structures of interest rates

One key purpose of an asset pricing model like the CAPM is to extract from market data what is the required rate of return for a given equity investment.

Assuming the yield curve embed investors' expectations of future short rates, it can be used as a benchmark for your own forecast.

- If you expect interest rates to fall more than what appears to be expected by the average investor, then you might increase the average maturity of your portfolio of fixed income securities expecting to obtain large capital gains while decreasing your reinvestment risk.
- This said, an expected increase (or decrease) in interest rates can be driven by expected changes in inflation, in the real rate of interest and/or in the liquidity premium.

The shape of the yield curve could also be interpreted as a business cycle signal.

- Long-term rates tend to rise in anticipation of an economic expansion.
- A downward-sloping curve sugests that yields are likely to fall, a signal of a coming recession.

Chapter 15 of the textbook

Learning Objectives covered

- L01 to L06

Concept checks

- Concept checks 1 to 9 (solutions provided at the end of the chapter).

Exercises

- Suggest solving 15-10 and 15-16.
- It is possible to extract the spot rates from coupon-paying bonds and then calculate the forwards (see extra material at the end of the slide deck and related Excel file).

15-10 Solution

Maturity	YTM@T=0	EYTM@T=1	Forward@T=0	Forecast@T=1
1	4%	5%	4.00%	6.01%
2	5%	6%	6.01%	7.01%
3	6%	7%	8.03%	

a)

Face value	100
Maturity@T=0	3
Price@T=0	83.96
Maturity@T=1	2
Price@T=1	89.00
Return T=1to2	6.00%

b) See above
Your expected YTM in one year is below the forecast derived from the forward rates.

Maturity	Price	YTM@T=0	Forward
1	94.34	6.00%	6.00%
2	84.99	8.47%	11.00%

a)

Face value	100
Coupon	12%
Maturity	2
Price@T=0	106.51

b) See above

c)

Price@T=1	100.90
HPR@T=1	6.00%

d) Er2<F2

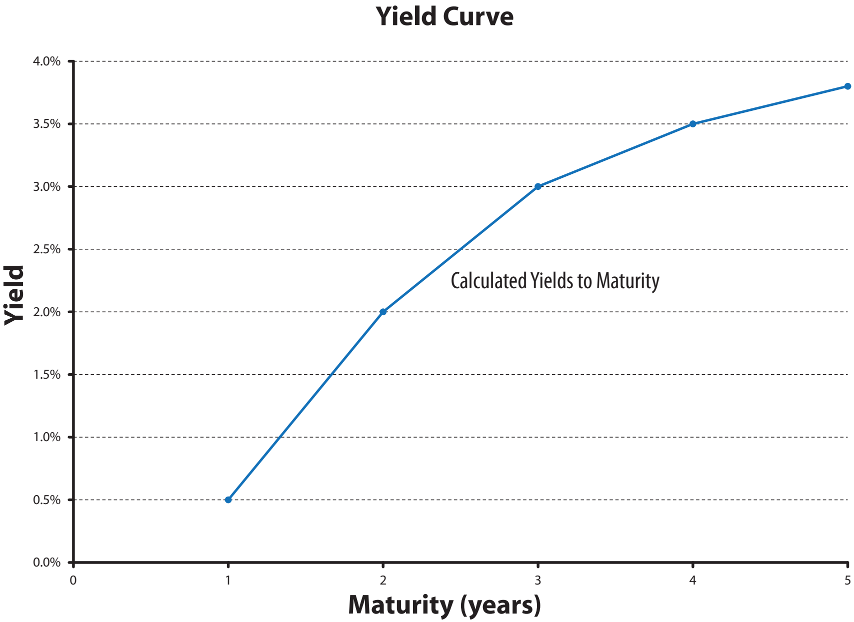
EHPR>6%

Calculation of the Theoretical Treasury Spot Rate Curve

-------------- Observed Data	--------------		Calculated
Maturity	Price	Coupon	Yield
1 year	99.50	0.0%	0.5%
2 years	103.88	4.0%	2.0%
3 years	100.00	3.0%	3.0%
4 years	101.84	4.0%	3.5%
5 years	103.13	4.5%	3.8%

$$B_0 = \sum_{t=1}^{n} \frac{CF_t}{(1+i)^t} \rightarrow 99.50 = \frac{100}{(1+i)^1} \rightarrow i = \frac{100}{99.50} - 1 = 0.5\%$$

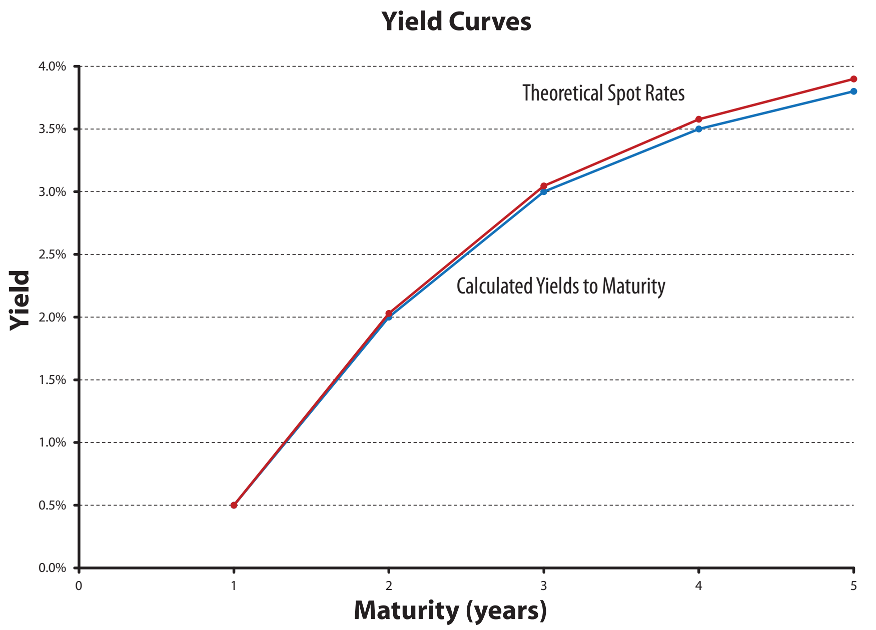
$$\rightarrow 103.88 = \frac{4}{(1+i)^1} + \frac{104}{(1+i)^2} \rightarrow i = 2.0\%$$


Yield Curve
Yield
4.0%
3.5%
3.0%
2.5%
2.0%
1.5%
1.0%
0.5%
0.0%
0
1
2
3
4
5
Maturity (years)
Calculated Yields to Maturity

-------------- Observed Data --------------			Calculated	Bootstrap
Maturity	Price	Coupon	Yield	Spot
1 year	99.50	0.0%	0.5%	0.50%
2 years	103.88	4.0%	2.0%	2.03%
3 years	100.00	3.0%	3.0%	3.05%
4 years	101.84	4.0%	3.5%	3.58%
5 years	103.13	4.5%	3.8%	3.90%

$$B_0 = \sum_{t=1}^{n} \frac{CF_t}{(1+z_t)^t} \rightarrow 103.88 = \frac{4}{(1+z_1)^1} + \frac{104}{(1+z_2)^2}$$

$$103.88 = \frac{4}{(1+.005)^1} + \frac{104}{(1+i_2)^2} \rightarrow i = \sqrt{\frac{104}{103.88-3.98}} - 1 = 2.03\%$$



Yield Curves
Theoretical Spot Rates
Calculated Yields to Maturity
Yield
Maturity (years)
4.0%
3.5%
3.0%
2.5%
2.0%
1.5%
1.0%
0.5%
0.0%
0
1
2
3
4
5

Forward Rates

${}_t f_m$ is the forward rate m periods from now, for t periods $(z_1 = {}_1 f_0)$.

Under some assumptions, investing in a two year bond shall yield the same as investing in a one year bond and rolling it into another one year bond at maturity:

$$(1+z_2)^2 = (1+z_1)(1+{}_1 f_1) \rightarrow {}_1 f_1 = \frac{(1+z_2)^2}{(1+z_1)} - 1$$

$$(1+z_3)^3 = (1+z_1)(1+{}_1 f_1)(1+{}_1 f_2) = (1+z_1)(1+{}_2 f_1)^2$$

In general

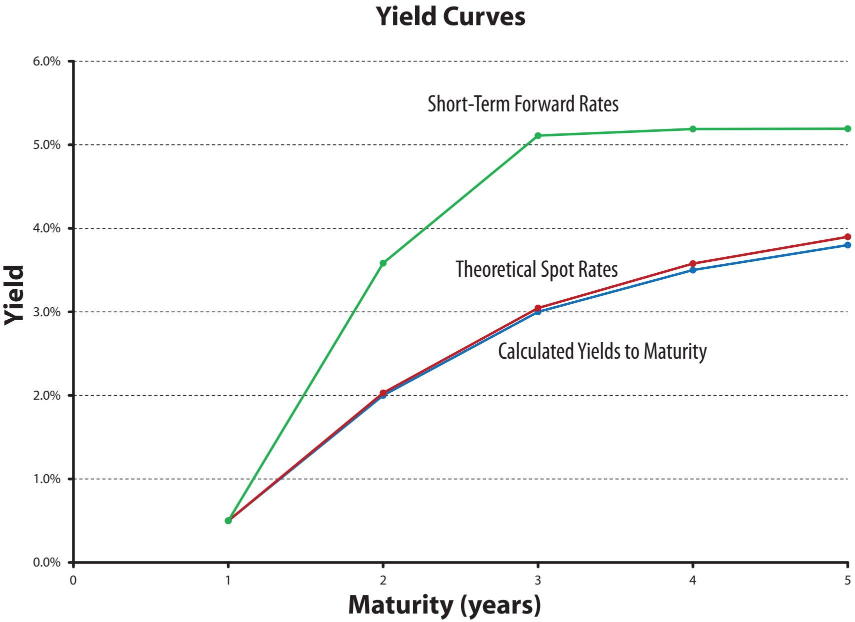
$$(1+z_{t+m})^{t+m} = (1+z_m)^m (1+{}_t f_m)^t \rightarrow {}_t f_m = \left[\frac{(1+z_{t+m})^{t+m}}{(1+z_m)^m}\right]^{1/t} - 1$$

$$Z_t = \left[\prod_{i=1}^{t} (1+{}_t f_m)\right]^{1/t} - 1 \approx \frac{\sum_{i=1}^{t} {}_t f_m}{t}$$

----------- Observed Data -----------			------------- Calculated ------------		
Maturity	Price	Coupon	Yield	Spot	Forward
1 year	99.50	0.0%	0.5%	0.50%	0.50%
2 years	103.88	4.0%	2.0%	2.03%	3.58%
3 years	100.00	3.0%	3.0%	3.05%	5.12%
4 years	101.84	4.0%	3.5%	3.58%	5.19%
5 years	103.13	4.5%	3.8%	3.90%	5.19%

$$ {}_1f_1 = \frac{(1+z_2)^2}{(1+z_1)} - 1 = \frac{(1+.0203)^2}{(1+.005)} - 1 = \frac{1.041}{1.005} - 1 = 3.58\% $$

$$ {}_1f_2 = \left[\frac{(1+z_{1+2})^{1+2}}{(1+z_2)^2}\right]^{1/1} - 1 = \frac{(1+.0305)^3}{(1+.0203)^2} - 1 = \frac{1.0943}{1.041} - 1 = 5.12\% $$



Yield Curves
Short-Term Forward Rates
Theoretical Spot Rates
Calculated Yields to Maturity
Yield
6.0%
5.0%
4.0%
3.0%
2.0%
1.0%
0.0%
0
1
2
3
4
5
Maturity (years)

Calculate the present value of a 3-year bond with an annual coupon of 3% using:

i) the above yield to maturity

$$B_{t=0} = \frac{3}{(1+.03)^1} + \frac{3}{(1+.03)^2} + \frac{103}{(1+.03)^3} = 100$$

ii) the above spot rates

$$B_{t=0} = \frac{3}{(1+.005)^1} + \frac{3}{(1+.0203)^2} + \frac{103}{(1+.0305)^3} = 100$$

iii) the above forward rates

$$B_{t=0} = \frac{3}{(1+.005)} + \frac{3}{(1+.005)(1+.0358)} + \frac{103}{(1+.005)(1+.0358)(1+.0512)} = 100$$

$$B_{t=0} = \frac{3 + \dfrac{3 + \dfrac{103}{(1+.0512)}}{(1+.0358)}}{(1+.005)} = 100$$