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Investments & Portfolio Management

Index Models and the Capital Asset Pricing Model

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Limitations of the Markowitz approach

The Markowitz approach requires the estimation (forecasting) of a large number of parameters.

- n returns, but n x n variance-covariance matrix... (50 securities require 1,325 estimates).
- Estimating the parameters from past data provides no guarantee that the parameters are internally consistent (a 'positive definite' matrix) or statistically significant.

The output of the Markowitz optimization is highly sensitive to the inputs.

- The asset allocation output can be extreme or counter-intuitive.
- Small changes in returns or variances/covariances can lead to large changes in allocation.

The approach does not provide a basis for forecasting returns or variances/covariances.

- No insight on the process driving the returns and the variances/covariances.
- It is simplistic to use past returns and past variances/covariances 'as is' (future = past ?).

$$r_i = Er_i + \beta_i m + e_i$$

r_i: the realized return of security i; Er_i: the expected return of security i (a 'mean')
m: a latent factor which influences all returns and induces correlations between securities
β_i: the sensitivity of the return of security i to the factor m
e_i: measures firm-specific surprises and has a zero expected value (**uncorrelated to** m)

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2$$

σ_i^2: the variance of the return of security i
$\beta_i^2 \sigma_m^2$: the variance of the return of security i, as induced by factor m (systematic)
$\sigma_{e_i}^2$: the variance of the return of security i, which is unique to security i (unsystematic)

$$\sigma_{ij}^2 = Cov\left(r_i, r_j\right) = Cov\left(\beta_i m + e_i, \beta_j m + e_j\right) = \beta_i \beta_j \sigma_m^2$$

σ_{ij}^2: the covariance between of the returns of security i and of security j

$$R_i = r_i - r_f \qquad R_M = r_M - r_f$$

R_i: the realized return of security i less the risk-free return (the 'excess return' of security i)
R_M: the realized return of index M less the risk-free return (the 'excess return' of the index M)

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t) \leftrightarrow ER_i = \alpha_i + \beta_i ER_M$$

Use above regression to estimate β_i; α_i is the abnormal return of security i (e.g. when R_M=0).
Use second equation to estimate ER_i using two parameters (α_i and β_i) + an estimate of ER_M.

$$\sigma_{ij}^2 = \rho_{ij}\sigma_i\sigma_j = \beta_i\beta_j\sigma_M^2 \leftrightarrow \rho_{ij} = \frac{\beta_i\beta_j\sigma_M^2}{\sigma_i\sigma_j} = \frac{\beta_i\beta_j\sigma_M^2}{\sigma_i\sigma_j}\frac{\sigma_M^2}{\sigma_M^2} = \frac{\beta_i\sigma_M^2}{\sigma_i\sigma_M}\frac{\beta_j\sigma_M^2}{\sigma_j\sigma_M} = \rho_{iM}\rho_{jM}$$

Estimates are: $\alpha_i, \beta_i, \sigma_{e_i}^2, ER_M, and\, \sigma_M^2$ (152 estimates rather than 1,325 for 50 securities!)

Assumptions

- Security returns follow normal distributions;
- The index used is a valid proxy of the latent factor.

Risk is either systematic or unsystematic (but what about industry risk?).

- The index model rules out correlations between residuals, possibly leading to sub-optimization versus the Markowitz approach.

When past data is just too old? (trade-off between statistical power and economic relevance).

The parameters $\alpha_i, \beta_i, \sigma^2_{e_i}$

- Estimated with error and could be subject to 'return to the mean' (mean revert);
- Can be influenced by future changes in relationships and by changes in the characteristics of the firm induced by management decisions (e.g. capital structure and business risks).

Using the single index model to create an equity portfolio

Select n individual securities and an index (n+1 securities)

- S&P500 (e.g. ETF) and six large US stocks (seven securities);
- The S&P500 index is used as proxy of the latent factor.

Estimate the index model parameters $(\alpha_i, \beta_i, \sigma^2_{e_i})$

- Get high-quality historical data (1983 to 2012 from WRDS) and run regressions.

Specify the optimal portfolio according to process illustrated in textbook (pages 271 to 280).

- Determine optimal weights of individual securities according to the ratio between alpha and standard deviation of the error terms to specify the actively-managed portfolio;
- Determine optimal weights of index and active actively-managed portfolio;
- Compare to Markovitz portfolio and predict performance of both portfolios.

In addition we will perform an out of sample test (using 2013 to 2017 data from WRDS).

Output of the OLS regression

Regression Statistics	
Multiple R	0.6908565
R Square	0.4772827
Adjusted R	0.4758226
Standard Er	0.051875
Observatior	360

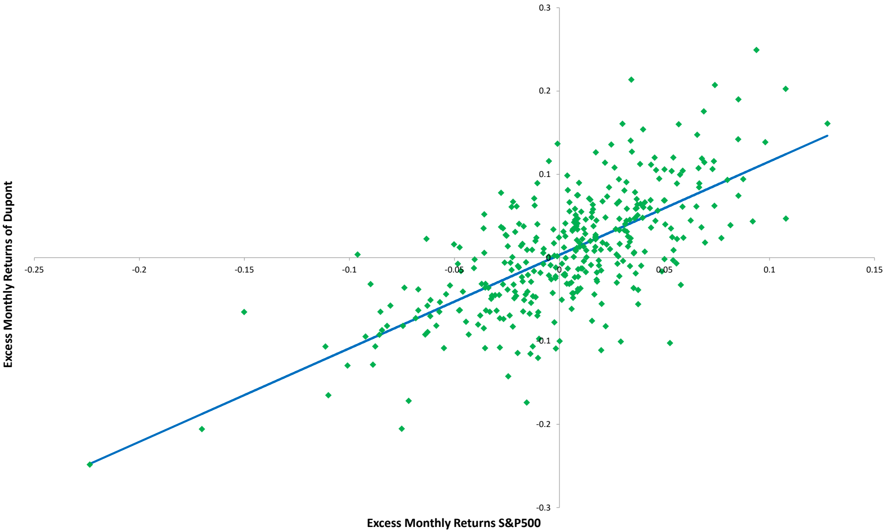
ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	0.8796473	0.8796473	326.88264	2.258E-52
Residual	358	0.9633847	0.002691		
Total	359	1.843032			

	Coefficients	*Standard Erro*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	0.0033104	0.0027449	1.2060039	0.2286124	-0.002088	0.0087087	-0.002088	0.0087087
X Variable 1	1.1218256	0.0620482	18.079896	2.258E-52	0.9998008	1.2438505	0.9998008	1.2438505

The high F-value indicates that the regression is statistically significant (good).
The medium P-value of the intercept indicates that α is not statistically significant (good if using CAPM).
The P-value of zero for the slope indicates that β is highly significant (good).
The point estimate of β is 1.12 (from 1.00 to 1.24 in a +/- 5% CI).

Security characteristic line of Dupont (1983 to 2012; 60 months)



	$\hat{\alpha}$	P-value	$\hat{\beta}$	P-value	Low 95%	Upper 95%	Adj. $\hat{\beta}$
ABT	0.007	0.012	0.611	0	0.484	0.739	0.741
CPB	0.006	0.097	0.620	0	0.471	0.769	0.747
CVX	0.006	0.025	0.721	0	0.601	0.840	0.814
DD	0.003	0.229	1.122	0	1.000	1.244	1.081
IBM	0.003	0.352	0.970	0	0.819	1.121	0.980
MMM	0.004	0.106	0.760	0	0.646	0.874	0.840

- The estimated α are economically small, mostly not statistically significant at the 5% level, and would therefore normally be ignored, but will nevertheless be used as predictors for the purpose of this exercise.
- The estimated β are in the neighborhood of 1 and highly statistically significant.
- The 5% confidence intervals for the estimated β are fairly large, suggesting that these estimates are with some error. Adj. $\hat{\beta} = 2\hat{\beta}/3 + 1/3$ to account for mean-reversion to 1.

	$\hat{\alpha}$	$\hat{\beta}$	Excess ER	σ Ex. ER	Sharpe Ratio	$\hat{\beta}$ 2013-17
S&P500	0.000	1	0.048	0.153	0.32	1
ABT	0.025	0.611	0.055	0.209	0.26	1.531
CPB	0.019	0.620	0.049	0.239	0.21	0.401
CVX	0.021	0.721	0.056	0.208	0.27	1.219
DD	0.011	1.122	0.066	0.248	0.26	1.697
IBM	0.011	0.970	0.058	0.267	0.22	0.935
MMM	0.014	0.760	0.051	0.204	0.24	1.096

- The Sharpe ratio of the index is higher than the individual stocks, which is to be expected for a 360-month period since the index is wells diversified.
- As a preview for the out-of sample analysis, the β estimated using a 360-month period are used as predicted β, but the actual β for the subsequent 60-month period are different.

Actual correlations between return residuals are not all zeros

	ABT	CPB	CVX	DD	IBM	MMM
ABT	1					
CPB	**0.207**	1				
CVX	-0.016	-0.062	1			
DD	-0.060	-0.016	0.122	1		
IBM	-0.134	-0.150	-0.069	-0.020	1	
MMM	-0.019	0.018	0.008	**0.327**	-0.001	1

- The zero correlation return residuals assumption of the index model does not hold.
- Therefore, there are significant differences between the covariance matrix of returns for the index model and the covariance matrix estimated statistically using the data directly.
- This and using alphas to determine weights for the actively managed model portfolio is likely to lead to differences between the index model portfolio and the Markowitz portfolio.

Weights of portfolios

	Index Model	Markowitz
S&P500	-1.82	0.39
ABT	0.70	0.18
CPB	0.39	0.12
CVX	0.66	0.19
DD	0.35	-0.14
IBM	0.22	0.06
MMM	0.50	0.20

- Consult the Excel file for the details regarding the computation of the weights of the two portfolios (pages 276 to 279 of the textbook).
- The weights of the Index Model are somewhat counter intuitive, but explained by the 'extra return' which is assumed from the $\hat{\alpha}$ for each individual stocks.

	Index Model		Markowitz	
	Expected	Actual	Expected	Actual
Excess Return	0.068	0.158	0.084	0.129
σ Excess Return	0.232	0.209	0.102	0.103
Sharpe Ratio	0.29	0.76	0.82	1.248

- The performance of the index model portfolio (expected and realized Sharpe ratio) is significantly lower than the Markowitz portfolio.
- This is an illustration of expecting to earn alpha out of $\hat{\alpha}$ which we knew were not statistically significant. We have to expect it would likely not turn out to work well.
- But it also illustrates that if you use weights to earn excess return from stock picking, you deviate from optimal diversification and there is a cost to do so. Otherwise, the discrepancy between the two approaches is likely to be much lower.

Asset pricing models

What we have previously developed is excellent, but still misses a few key considerations.

What would happen if:

- rather than focusing on a single investor, we would take into consideration all investors simultaneously;
- we use the fact that no more and no less that all securities have to be owned at all time.

Also we still missing a framework to value assets, especially stocks.

What would happen if:

- a security is overvalued or undervalued (i.e. an apparent arbitrage opportunity).

By using a set of assumptions, we can further develop the Index Model into the Capital Asset Pricing Model (CAPM), and address the above noted issues in an useful equilibrium manner.

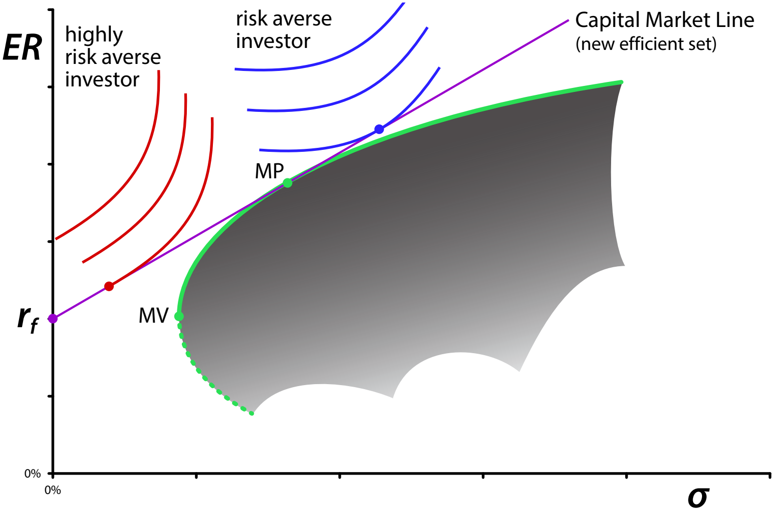
The two key assumptions (in addition to the usual technical ones)

- All investors are rational **mean-variance optimizers** (only mean and variance matters to them: prefer maximum return for given volatility and minimum variance for given return).
- All investors have **identical** estimates of expected return, variances, and covariances for every security (all relevant information is publicly available $\rightarrow$ homogeneous expectations).

Implications

- All investors therefore have the **same efficient frontier** of risky assets (without R_f).
- With R_f, all investors see as **optimal** to combine R_f and the market portoflio MP.
- Therefore, all investors hold risky securities in the **same proportions**, i.e. in the proportions of the Market Portfolio.
- The Sharpe ratio of MP reflect the risk-aversion of an 'average' investor.
- The **expected return of any asset** in MP is such that the extra return from holding the asset in the portfolio exactly **compensates for the extra variance** of holding the asset.

Market equilibrium according to CAPM: the graph



We previously derived y^*, the optimal proportion an investor allocates to the risky portfolio.

$$y^* = \frac{Er_p - r_f}{A\,\sigma_p^2}$$

However, if we take into consideration all investors under the lens of a representative agent (i.e. the 'average investor'), all lending and borrowing cancel out.

- All the money lent equals all the money borrowed and cancel each other on a net basis.
- Therefore, y^* has to equal to one economy-wide and for the representative agent.

$$y^* = 1 = \frac{Er_M - r_f}{\overline{A}\,\sigma_M^2} \rightarrow \overline{A} = \frac{Er_M - r_f}{\sigma_M^2} \rightarrow Er_M - r_f = \overline{A}\sigma_M^2$$

- The risk-return trade-off (i.e. the compensation received from bearing the market risk) is determined by the average degree of risk aversion across all investors!

If you invest a little in security i by borrowing, the expected return of the portfolio increases:

$$\triangle r_p = \triangle \left(Er_i - r_f \right)$$

Meanwhile, the variance of the portfolio also increases:

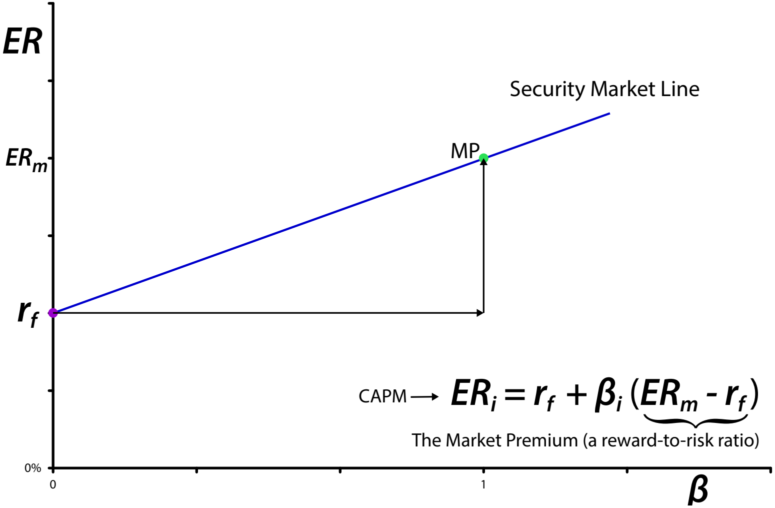
$$\triangle \sigma_p^2 = \triangle Cov \left(r_i, r_M \right) = \triangle \sigma_{i,M}$$

But the risk-return of that investment has to be identical to the market risk-return trade-off.

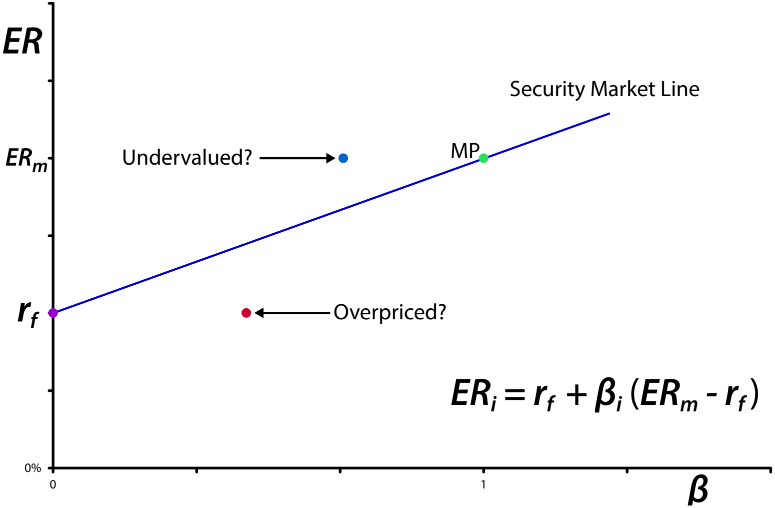
$$\frac{\triangle \left(Er_i - r_f \right)}{\triangle \sigma_{i,M}} = \frac{Er_M - r_f}{\sigma_M^2} \rightarrow Er_i = r_f + \frac{\sigma_{i,M}}{\sigma_M^2} \left(Er_M - r_f \right) = r_f + \frac{\rho_{iM} \sigma_i}{\sigma_M} \left(Er_M - r_f \right)$$

$$= r_f + \beta_i \left(Er_M - r_f \right) \rightarrow \beta_i = \frac{\sigma_{i,M}}{\sigma_M^2} = \frac{\rho_{iM} \sigma_i}{\sigma_M}$$

Which is known as the CAPM! Replicating the market portfolio (a passive strategy) is efficent!



$$\text{CAPM} \longrightarrow ER_i = r_f + \beta_i \, (\underbrace{ER_m - r_f}_{\text{The Market Premium (a reward-to-risk ratio)}})$$


ER
ERm
rf
0%
0
1
β
Security Market Line
MP
Undervalued?
Overpriced?
$ER_i = r_f + \beta_i (ER_m - r_f)$

Roll's critique

In 1977, Richard Roll got "A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory." published in JFE.

The article is a well-thought out critique of the CAPM, basically suggesting that the CAPM cannot be satisfactorily empirically tested.

- There is a mean-variance tautology at the core of the model as testing the CAPM is (mathematically) equivalent to testing the mean-variance efficiency of the proxy portfolio used, while the Market Portfolio is assumed to be mean-variance efficient.
- The theory assumes the Market Portfolio to include all assets in existence. But the returns on many assets classes are unobservable. So the 'true' Market Portfolio is unobservable.

It is not possible to determine if the unobservable Market Portfolio is mean-variance efficient (same with any partition as a sub-porfolio), making CAPM not 'truly' testable.

Also, CAPM is a model of expected returns which are also not readily available, and therefore need to be proxied using actual/realized returns or otherwise forecasted.

Concept checks

- Chapter 8: concept checks 1 to 5 (solutions provided at the end of the chapter).
- Chapter 9: concept checks 1 to 5 (solutions provided at the end of the chapter).

Exercises

- Chapter 8: suggest 8-4, and 8-13.
- Chapter 9: suggest 9-20, and 9-23.
- Solutions follow next slides, and Excel solution file in available in D2L.

Stock	Alpha	Beta	R2	SD res
A	1%	1.2	0.576	10.30%
B	-2%	0.8	0.436	9.10%

a. Firm-specific risk is measured by the residual standard deviation.
   Stock A has more firm-specific risk: 10.3% > 9.1%.

b. Market risk is measured by beta, the slope coefficient of the regression.
   Stock A has a larger beta coefficient: 1.2 > 0.8.

c. R2 measures the fraction of total variance of return explained by the market return.
   A's R2 is larger than B's: 0.576 > 0.436.

d. The average rate of return in *excess* of that predicted by the sn index model is measured by alpha, the intercept of the SCL. aA = 1% is larger than aB = –2%.
   Rewriting the SCL equation in terms of total return (r) rather than excess return (R):
   $r_A - r_f = \alpha + \beta \times (r_M - r_f) \Rightarrow r_A = \alpha + r_f \times (1 - \beta) + \beta \times r_M$
   The intercept is now equal to:
   $\alpha + r_f \times (1 - \beta) = 1\% + r_f \times (1 - 1.2)$
   Since rf = 6%, the intercept would be: $1\% + 6\%(1 - 1.2) = 1\% - 1.2\% = -0.2\%$

8-13 Solution

Stock	ER	Beta	SD
A	11%	0.8	10%
B	14%	1.5	12%
Index	12%		
Rf	6%		

$$\alpha_A = r_A - \left[r_f + \beta_A \times \left(R_M - r_f\right)\right] = .11 - [.06 + .8 \times (.12 - .06)] = 0.002$$

$$\alpha_B = r_B - \left[r_f + \beta_B \times \left(R_M - r_f\right)\right] = .14 - [.06 + 1.5 \times (.12 - .06)] = -0.01$$

Stock A would be a good addition to a well-diversified portfolio.
A short position in stock B may be desirable.

Advisor	Return	Beta
A	19%	1.5
B	16%	1.0

a. Stock picking performance is revealed by abnormal return (i.e. realized alpha, the difference between the actual return and the predicted return using CAPM).
Without Rf and Rm, we cannot calculate the predicted return.

b. Rf = 6% and Rm=14%
$\alpha_A = r_A - [r_f + \beta_A \times (R_M - r_f)] = .19 - [.06 + 1.5 \times (.14 - .06)] = 0.01$
$\alpha_B = r_B - [r_f + \beta_B \times (R_M - r_f)] = .16 - [.06 + 1.0 \times (.12 - .06)] = 0.02$
Investor B has the larger abnormal return and appears to be the superior stock selector.

c. Rf = 3% and Rm=15%
$\alpha_A = r_A - [r_f + \beta_A \times (R_M - r_f)] = .19 - [.03 + 1.5 \times (.15 - .03)] = -0.02$
$\alpha_B = r_B - [r_f + \beta_B \times (R_M - r_f)] = .16 - [.03 + 1.0 \times (.15 - .03)] = 0.01$
Investor B has the larger abnormal return and appears to be the superior stock selector.
Investor A having a negative abnormal return appears to have no selection ability.

9-23 Solution

	ER	Beta
Fund	14%	0.8
Market	15%	
Rf	5%	

a. $\alpha_A = r_A - \left[r_f + \beta_A \times \left(R_M - r_f\right)\right] = .14 - [.05 + .8 \times (.15 - .05)] = 0.01$
You should consider investing in this fund because alpha is positive, but have to ask yourself if it is superior selection ability or simply good luck?

b. $\beta_A = W_M \beta_M = W_M \times 1 \rightarrow W_M = \beta_A = 0.8$
$Er_A = W_f r_f + W_M r_M = 0.2 \times .05 + 0.8 \times .15 = 0.13$
$\alpha_A = r_A - Er_A = 0.14 - 0.13 = 0.01$